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Classes of Inverse Semirings and its Ordering

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Abstract- In this paper we study the conditions under which an additively inverse semiring is a band and commutative. Also in the case of totally ordered additive inverse semiring in which $(S,+)$ is positively totally ordered, we prove the additive structure is $a+b=b+a=\max(a,b)$ for all a,b in S. We have also framed an example for this result by considering 4 elements.

Index Terms - Additive inverse semiring; zero-square semiring; positively totally ordered semiring.

1. INTRODUCTION

Additive and multiplicative structures play an important role in determining the structure of semiring. If the multiplicative structure of a semiring is a rectangular band, then its additive structure is a band. Karvellas [4] proved that if the additively inverse semiring contains right multiplicative identity, then its additive structure is commutative. In [9] Zeleznekow studied additive inverse semirings and examined the conditions under which $(S,+)$ is a semilattice. In this paper we study the conditions under which $(S,+)$ is a commutative band and determine the structure of additively inverse semiring [4], [8] if $(S,+)$ is positively totally ordered [1],[2], [5],[6].

2. PRELIMINARIES

In this section, we have given some important definitions that are used in the theorems which are stated and proved in the following sections.

Definition 2.1

A triple $(S,+,\cdot)$ is called a semiring if $(S,+)$ is a semigroup; $(S,\cdot)$ is a semigroup; $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for every a,b,c in S. The two operations + and · are called additive and multiplicative operations respectively.

Definition 2.2

A semiring $(S,+,\cdot)$ is called an additive inverse semiring if $(S,+)$ is an inverse semigroup that is for each $a\in S$ there exists a unique element $a'\in S$ such that $a+a'+a=a$ and $a'+a+a'=a'$.

Definition 2.3

In a totally ordered semiring $(S,+,\cdot,\leq)$

(i) $(S,+,\leq)$ is positively totally ordered (p.t.o), if $a+x\geq a,x$ for all a,x in S.

(ii) $(S,\cdot,\leq)$ is positively totally ordered (p.t.o), if $ax\geq a,x$ for all a,x in S.

(iii) $(S,+,\leq)$ is negatively totally ordered (n.t.o), if $a+x\leq a,x$ for all a,x in S.

(iv) $(S,\cdot,\leq)$ is negatively totally ordered (n.t.o) if $ax\leq a,x$ for all a,x in S.

Definition 2.4

A semiring S is Positive Rational Domain (PRD) if $(S,\cdot)$ is an abelian group.

Definition 2.5

Zeroid of a semiring $(S,+,\cdot)$ is the set of all x in S such that $x+y=y$ or $y+x=y$ for some y in S. we may also term this as the zeroid of $(S,+)$ and it is denoted by Zd.

Definition 2.6

A semiring S with multiplicative zero is said to be a zero square semiring if $a^2=0$ for all a in S.

Definition 2.7

A semigroup $(S,+)$ is commutative if $a+x=x+a$ for all a,x in S.

A semigroup $(S,\cdot)$ is commutative if $ax=xa$ for all a,x in S.

Definition 2.8

A semigroup $(S,+)$ is additively idempotent if $a+a=a$ for all a in S.

Definition 2.9

An element a in a semigroup $(S,+)$ is periodic if $ma=na$ where m and n are positive integers. A semigroup $(S,+)$ is periodic if every one of its elements is periodic.

Definition 2.10

A semigroup $(S,+)$ is band if $a+a=a$ for all a in S

A semigroup $(S,\cdot)$ is band if $a^2=a$ for all a in S

Definition 2.11

A semiring S is said to be mono semiring if

$a+b=ab$ for all a,b in S.

Note: Undefined concepts refer [3]

3. STRUCTURE OF ADDITIVE INVERSE SEMIRINGS

Theorem 3.1. *Let $(S,+,\cdot)$ be an additively inverse PRD. Then $(S,+)$ is a commutative band.*

Proof. Let $a \in S$. Then there exists $a' \in S$
Such that $a+a'+a=a$
$\Rightarrow a+a'+a+a'=a+a'$
$\Rightarrow (a+a')\cdot 1+(a+a')\cdot 1=a+a'$
$\Rightarrow (a+a')(1+1)=(a+a')\cdot 1$
$\Rightarrow 1+1=1$, since $(S,\cdot)$ is a group.
$\Rightarrow b(1+1)=b.1$ for all $b \in S$
$\Rightarrow b+b=b$
Therefore $(S,+)$ is a band
Similarly by taking $a'+a+a'=a'$, we can prove that $(S,+)$ is a band.
Using theorem 4(1) of [4], $(S,+)$ is a commutative.
Hence $(S,+)$ is commutative band.

Theorem 3.2. *Let $(S,+,\cdot)$ be an additive inverse semiring. Then the zeroid of $(S,+)=Zd=\{a+a'/a\in S\}$ is a multiplicative ideal.*

Proof. $a=a+(a'+a)$ also $a'=a'+(a+a')$
Therefore $a'+a$ and $a+a' \in Zd$

Now we prove that Zd is a multiplicative ideal.
Let $s \in S$
$a=a+a'+a$ implies
$sa=sa+sa'+sa$
$=sa+s(a'+a)$
$\Rightarrow s(a'+a)\in Zd$
Also $a'=a'+a+a'$ implies
$sa'=sa'+sa+sa'$
$=sa'+s(a+a')$
$\Rightarrow s(a+a')\in Zd$
Similarly $(a+a')s \in Zd$
Therefore Zd is a multiplicative ideal.

Theorem 3.3. *Let S be an additive inverse semiring. If $(S,\cdot)$ is a band, then aa' and $a'a$ are additively periodic elements.*

Proof. Given that S is an additive inverse semiring
$a+a'+a=a \Rightarrow a^2+aa'+a^2=a^2$
since $(S,\cdot)$ is a band then $a^2=a,\ a'^2=a'$ for all $a\in S$
$a+aa'+a=a \Rightarrow aa'+aa'^2+aa'=aa'$
$\Rightarrow aa'+aa'+aa'=aa' \Rightarrow 3aa'=aa'$
Thus aa' is additive periodic
Also $a'+a+a'=a' \Rightarrow a'^2+a'a+a'^2=a'^2$
$\Rightarrow a'+a'a+a'=a' \Rightarrow a'a+a'a^2+a'a=a'a$
$\Rightarrow a'a+a'a+a'a=a'a \Rightarrow 3a'a=a'a$
Thus $a'a$ is additive periodic.

Theorem 3.4. If S is an additive inverse semiring and $(s,\cdot)$ is rectangular band, then following identities are true.
(i) $a+a'a^2+a=a$
(ii) $aa'+a'+aa'=aa'$
(iii) $a'^2a+a'a^2+a'^2+a=a'^2a$

Proof. Given that S is an additive inverse semiring
$a+a'+a=a$
$a^2+a'a+a^2=a^2$
$a^3+a'a^2+a^3=a^3$
Since $(s,\cdot)$ is rectangular band
$a+a'a^2+a=a$
By using above condition (i)
$\Rightarrow aa'+a'a^2a'+aa'=aa'$
$aa'+a'+aa'=aa'$
By using above condition (i)
$\Rightarrow a'a+a'^2a^2+a'a=a'a$
$a'^2a+a'^3a^2+a'^2a=a'^2a$

4. STRUCTURE OF TOTALLY ORDERED SEMIRINGS

Theorem 4.1. *If S is a totally ordered additive inverse semiring and $(S,+)$ is positively totally ordered (p.t.o), then*

(i) $a = a'$ *for all* $a \in S$

(ii) $(S,+)$ *is a band.*

(iii) $a+b = b+a = \max(a,b)$.

Proof. (i) Since S is an additive inverse semiring for each $a \in S$ there exists $a' \in S$ such that
$a = a + a' + a \geq a'$,
since $(S,+)$ is p.t.o (1)
Also $a' = a' + a + a' \geq a + a' \geq a$ (2)
Using (1) and (2) $a = a'$ for all $a \in S$.

(ii) $a = a + a' + a \geq a + a' \geq a$
since $(S,+)$ is p.t.o
Therefore $a = a + a'$ (3)
Now $a = a + a' + a = a + a$ using (3)
Hence $(S,+)$ is a band.

(iii) Let $a, b \in S$
Suppose $a < b$
Then $a + a \leq a + b \leq b + b$
$\Rightarrow a \leq a + b \leq b$
since $(S,+)$ is a band. (4)
From (4) $a + b \leq b$ (5)
Since $(S,+)$ is p.t.o, $a + b \geq b$ (6)
From (5) and (6) $a + b = b = \max(a,b)$
Also $a < b$ implies $a + a \leq b + a \leq b + b$
$\Rightarrow a \leq b + a \leq b$ (7)
Since $(S,+)$ is p.t.o, $b + a \geq b$ (8)
From (7) and (8) $b + a = b$.
Similarly we can prove that $a + b = b + a = \max(a,b)$ if $b < a$.

Theorem 4.2. *If S is a totally ordered multiplicative inverse semiring and $(S,\cdot)$ is positively totally ordered, then*

(i) $a = a^*$ *for all* $a \in S$

(ii) $(S,\cdot)$ *is a band.*

(iii) $ab = ba = \max(a,b)$.

Proof. (i) Since S is an additive inverse semiring for each $a \in S$ there exists $a' \in S$ such that
$a = aa^*a$ and $a = a^*aa^*$
now $a = aa^*a \geq aa^* \geq a^*$, since $(S,\cdot)$ is p.t.o. (9)
also $a^* = a^*aa^* \geq a^*a \geq a$ (10)
using (9) and (10) $a = a^*$ for all $a \in S$

(ii) $a = aa^*a = aa = a^2$
hence $(S,\cdot)$ is a band

(iii) $a, b \in S$
Suppose $a < b$
then $a^2 \leq ab \leq b^2$
$\Rightarrow a \leq ab \leq b$ since $(S,\cdot)$ is a band (11)
$ab \geq b$ (12)
from (11) and (12) $ab = b = \max(a,b)$
also $ba = b = \max(a,b)$
similarly we can prove $ab = ba = \max(a,b)$ if $b < a$

Proposition 4.3. *Let $(S,+,\cdot)$ is doubly inverse semiring. If $(S,+)$ and $(S,\cdot)$ are positively totally ordered, then $a+b = b+a = ab = ba = \max(a,b)$ and hence S is mono semiring.*

Proof. In the above theorem
$a + b = b + a = \max(a,b)$
Since $(S,\cdot)$ is an inverse semiring and p.t.o.
$a = aa'a \geq aa' \geq a$ (13)
Also $a' = a'aa' \geq a'a \leq a$ (14)
From (13) and (14) $a = a'$
Also $a = aa'a \geq aa' \geq a$
Therefore $a = aa' = a^2$ since $a' = a$
Suppose $a < b$
$\Rightarrow a^2 \leq ab \leq b^2 \Rightarrow a \leq ab \leq b$ (15)
Since S is p.t.o $ab \geq b$ (16)
From (15) and (16) $ab = b = \max(a,b)$
Then if $b < a$, we compare that $ab = a = \max(a,b)$

Theorem 4.4. *Let $(S,+,\cdot)$ be a totally ordered additive inverse semiring. If S contains the multiplicative identity, then $(S,+)$ is a commutative band.*

Proof. Using proposition 1 of [7]

$(S,+)$ is non-negatively ordered or non-positively ordered.
Suppose $(S,+)$ is non-negatively ordered
$a = a + a' + a = a + a + a' \geq a + a'$ by using theorem 4 (1) of [4]
Also $a' = a' + a + a' = a + a' + a' \geq a + a'$
$a + a' \leq a$ and $a + a' \leq a'$ (17)
From (11) $a + a' \leq a$
$\Rightarrow a' + a + a' \leq a' + a = a + a'$, since $(S,+)$ is commutative
$\Rightarrow a' \leq a + a'$ (18)
From (17) and (18), $a + a' = a$
$a = a + a' + a = a + a$
Therefore $(S,+)$ is a band.

Similarly we can prove the theorem if $(S,+)$ is non-positively ordered.

Example 4.5. *The following example $\{i, j, k, l\}$ with $l < k < j < i$ satisfies the conditions of theorem 4.1 in which $(S,+)$ is p.t.o and S is also a mono semiring. This example also satisfies the conditions of theorem 4.4.*

+	i	j	k	l
i	i	j	k	l
j	j	j	k	l
k	k	k	k	l
l	l	l	k	l

.	i	j	k	l
i	i	j	k	l
j	j	j	k	l
k	k	k	k	l
l	l	l	l	l

Theorem 4.6. *If S satisfies the conditions of theorem 4.1 and S is a zero square semiring with additive identity 0, then $S^2 = \{0\}$.*

Proof. By (iii) of theorem 4.1
$a + b = b + a = \max(a,b)$
Suppose $a < b$
Then $a + b = b + a = b$
$a + b = b$ implies
$\Rightarrow (a+b)b = b \cdot b$
$\Rightarrow ab + b^2 = b^2$
$\Rightarrow ab + 0 = 0$
Therefore $ab = 0$
Also $b(a+b) = b \cdot b$
$\Rightarrow ba + b^2 = b^2$
$\Rightarrow ba + 0 = 0$
Therefore $ba = 0$
Hence $S^2 = \{0\}$

Example 4.7. *The below tables satisfies the conditions of the theorem 4.6 with $m < l < k < j < i$.*

+	i	j	k	l	m
i	i	i	i	i	i
j	i	j	j	j	j
k	i	j	k	k	k
l	i	j	k	l	l
m	i	j	k	l	m

.	i	j	k	l	m
i	m	m	m	m	m
j	m	m	m	m	m
k	m	m	m	m	m
l	m	m	m	m	m
m	m	m	m	m	m

Theorem 4.8. *Let $(S,+,\cdot)$ be a totally ordered additively inverse semiring. If $(S,\cdot)$ is positively totally ordered, then the following are true.*
(i) $(ab)' \geq a$ *and* $(ab)' \geq b$
(ii) $ab \geq a'$ *and* $ab \geq b'$
(iii) $(a+b)' \leq (a+a)b$
(iv) $(a+b)' \leq ab$ *if* $(S,+)$ *is a band.*

Proof.
(i) $(ab)' = ab' \geq a,$ since $(S,\cdot)$ is p.t.o.
$(ab)' = a'b \geq b,$ since $(S,\cdot)$ is p.t.o.

(ii) $ab = a'b' \geq a'$ and $ab = a'b' \geq b'$, since $(S,\cdot)$ is p.t.o.

(iii) $(a+b)' = b' + a' \leq ab + ab = (a+a)b,$ using (ii)

(iv) From (iii) $(a+b)' \leq (a+a)b$ if $(S,+)$ is a band
$a+a=a$ hence $(a+b)' \leq ab$.

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